A drift estimation procedure for stochastic differential equations with additive fractional noise

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Joint work with Fabien Panloup & Samy Tindel.

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Outline

1. Introduction
   - Fractional Brownian motion
   - Ergodicity of fractional SDEs and approximation of stationary regime
   - Overview on drift estimation for fractional diffusion.

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Definition

Let $H \in (0,1)$. The d-dimensional *fractional Brownian motion* (fBm) with Hurst parameter H, denoted by $(B_t)_{t\geq 0}$, is a centered Gaussian process with covariance function given by :

$$\mathbb{E}[B_t^i B_s^j] = \frac{1}{2}\delta_{ij}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right] \qquad \text{for all } t, s \geq 0.$$

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- Stationary increments :

$$\mathbb{E}\left[(B_t^i - B_s^i)(B_t^j - B_s^j)\right] = \delta_{ij}|t-s|^{2H}.$$

- Self-similarity :

$$\mathcal{L}((B_{ct})_{t\geq 0}) = \mathcal{L}(c^H (B_t)_{t\geq 0}) \qquad \text{for all } c > 0.$$

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Remarks

▷ The fBm is neither a semimartingale nor a Markov process except for $H = 1/2$. In that case, B is the standard Brownian motion and has independent increments.

▷ Regularity: a.s. locally Hölder for all $\beta < H$.

Let $(W_t)_{t\in\mathbb{R}}$ be a standard Brownian motion.

Proposition (Mandelbrot Van Ness representation)

$$B_t := \int_{\mathbb{R}} (t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \mathrm{d}W_s, \quad t \in \mathbb{R},$$

where $x_+ = \max(0, x)$.

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Proposition (Volterra representation)

$$B_t := \int_0^t K_H(t,s) \mathrm{d}W_s, \quad t \geqslant 0,$$

where K_H is the deterministic kernel given by

$$K_H(t,s) = c_H \left[\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) \int_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}} (u-s)^{H-\frac{1}{2}} \mathrm{d}u \right].$$

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SDE with additive fractional noise:

$$dY_t = b(Y_t)dt + \sigma dB_t. \qquad \textbf{(E)}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma \in \mathbb{M}_{d\times d}$

Hairer (2005):

- Homogeneous Markovian structure: $(Y_t, (W_s)_{s\le t})_{t\ge 0}$ (state space: $\mathbb{R}^d \times \mathcal{W}$).
- Existence of invariant distribution: $\mu_\star \in \mathcal{M}_1(\mathbb{R}^d \times \mathcal{W})$.
- Uniqueness of $\mu_\star$ and rate of convergence in total variation distance: $t^{-\alpha_H}$ with

$$\alpha_H = \begin{cases} H(1-2H) & \text{if } H \in (0, 1/4] \\ 1/8 & \text{if } H \in (1/4, 1) \setminus \left\{\frac{1}{2}\right\} \end{cases} .$$

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Multiplicative case:

- Fontbona, Panloup (2017): $H \in (1/2, 1)$.
- Deya, Panloup and Tindel (2019): $H \in (1/3, 1/2)$.

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Euler scheme for a fixed step $\gamma > 0$: $Z_0 = Y_0$,

$$Z_{(n+1)\gamma} = Z_{n\gamma} + \gamma b(Z_{n\gamma}) + \sigma(B_{(n+1)\gamma} - B_{n\gamma}). \qquad (\mathbf{E}_\gamma)$$

Theorem (Cohen, Panloup '11) (Cohen, Panloup, Tindel '14)

$$\lim_{\gamma \to 0} \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{Z_{k\gamma}} = \bar{\mu}_\star \quad a.s.$$

in the sense of weak convergence on $\mathcal{M}_1(\mathbb{R}^d)$.

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1. On fractional Ornstein-Ulhenbeck processes (fOU) : $b_\vartheta(x) = -\vartheta x$.
   - Le Breton (1998)
   - Klepstyna, Le Breton (2002)
   - Hu, Nualart (2010)
   - Belfadi, Es-Sebaiy, Ouknine (2011)

2. On linear drift : $b_\vartheta(x) = \vartheta b(x)$
   - Tudor, Viens (2007)

3. Non parametric estimation in dimension 1
   - Comte, Marie (2018)

$\Longrightarrow$ *Most of them : continuous-time observation of the process.*

4. Discrete-time observations : Neuenkirsh, Tindel (2014)
   - $b_\vartheta(x) = \nabla F(x, \vartheta)$.
   - $\bar{Y}_0$ stationary solution : $\mathbb{E}\left[|b_{\vartheta_0}(\bar{Y}_0)|^2\right] = \mathbb{E}\left[|b_\vartheta(\bar{Y}_0)|^2\right] \quad \Leftrightarrow \quad \vartheta = \vartheta_0$.

2. Model and contruction of the estimators

Let Y be an $\mathbb{R}^d$-valued process such that $Y_0 = y_0$ and

$$dY_t = b_{\vartheta_0}(Y_t)\,dt + \sigma dB_t \qquad (\mathbf{E}_{\vartheta_0})$$

where $\sigma \in \mathbb{M}_{d\times d}$ is an invertible matrix, $\vartheta_0 \in \Theta$ is the unknown parameter and $\{b_\vartheta(.) \mid \vartheta \in \Theta\}$ is a known family of functions.

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$(\mathbf{H_0})$: $\Theta \subset \mathbb{R}^q$ is compact.

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$(\mathbf{C_w})$: We have $b \in \mathcal{C}^{1,1}(\mathbb{R}^d \times \Theta; \mathbb{R}^d)$ and:

(i) $\forall x, y \in \mathbb{R}^d$, $\forall \vartheta \in \Theta$,

$$\langle b_\vartheta(x) - b_\vartheta(y), x - y\rangle \leq \beta - \alpha|x-y|^2 \quad \text{and} \quad |b_\vartheta(x) - b_\vartheta(y)| \leq L|x-y|$$

(ii) $\forall x \in \mathbb{R}^d$, $\forall \vartheta \in \Theta$,

$$|\partial_\vartheta b_\vartheta(x)| \leq C\left(1 + |x|^r\right).$$

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Observations : $\{Y_{t_k} \mid 0 \leq k < n\}$ and $t_{k+1} - t_k = \kappa > 0$.

$$
\begin{aligned}
(\mathbf{C_w}) \quad & \Rightarrow (\mathbf{E}_\vartheta) \text{ admits a unique invariant distribution} \\
& \Rightarrow \nu_\vartheta \in \mathcal{M}_1(\mathbb{R}^d) \text{ (marginal invariant distribution).}
\end{aligned}
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$$(\mathbf{C_w}) \quad \Rightarrow (\mathbf{E}_\vartheta) \text{ admits a unique invariant distribution}$$
$$\Rightarrow \nu_\vartheta \in \mathcal{M}_1(\mathbb{R}^d) \text{ (marginal invariant distribution).}$$

Identifiability assumptions: denote by d a distance on $\mathcal{M}_1(\mathbb{R}^d)$.

$(\mathbf{I_w})$: We have: $\quad d(\nu_\vartheta, \nu_{\vartheta_0}) = 0 \quad \Leftrightarrow \quad \vartheta = \vartheta_0.$

$(\mathbf{I_s})$: There exists a constant $C > 0$ and a parameter $\varsigma \in (0,1]$ such that:

$$\forall \vartheta \in \Theta, \quad d(\nu_\vartheta, \nu_{\vartheta_0}) \geq C|\vartheta - \vartheta_0|^\varsigma.$$

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<u>**Approximation of ν_{ϑ}**</u>: Euler scheme associated to $(\mathbf{E}_\vartheta)$: $Z_0^\vartheta = y_0$

$$\forall k \geq 0, \quad Z_{s_{k+1}}^\vartheta = Z_{s_k}^\vartheta + (s_{k+1} - s_k) b_\vartheta(Z_{s_k}^\vartheta) + \sigma\left(B_{s_{k+1}} - B_{s_k}\right).$$

where $s_0 = 0$ and (s_k) is an increasing sequence such that $\lim_{k\to+\infty} s_k = +\infty$.

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- Constant step Euler scheme $Z^{\vartheta,\gamma}$: $s_k = k\gamma$.
- Decreasing step Euler scheme Z^ϑ: $\gamma_k := s_k - s_{k-1}$ is a decreasing sequence and $\lim_{k\to+\infty} \gamma_k = 0$.

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$$\forall k \geq 0, \quad Z^\vartheta_{s_{k+1}} = Z^\vartheta_{s_k} + (s_{k+1} - s_k)b_\vartheta(Z^\vartheta_{s_k}) + \sigma\,(B_{s_{k+1}} - B_{s_k}).$$

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Estimators:

$$\hat{\vartheta}_{N,n,\gamma} = \underset{\vartheta\in\Theta}{\operatorname{argmin}}\; d\left(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{Y_{t_k}},\; \frac{1}{N}\sum_{k=0}^{N-1}\delta_{Z^{\vartheta,\gamma}_{k\gamma}}\right),$$

$$\hat{\vartheta}_{N,n} = \underset{\vartheta\in\Theta}{\operatorname{argmin}}\; d\left(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{Y_{t_k}},\; \frac{1}{s_N}\sum_{k=0}^{N-1}\gamma_{k+1}\delta_{Z^{\vartheta}_{s_k}}\right).$$

where $d \in \mathcal{D}_p := \{\text{distances } d \text{ on } \mathcal{M}_1(\mathbb{R}^d);\, \exists\, c > 0, \forall \nu,\mu,\; d(\nu,\mu) \leq c\mathcal{W}_p(\nu,\mu)\}$

3 Consistency results

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Theorem (Panloup, Tindel, V. '19)

Assume $(\mathbf{H_0})$, $(\mathbf{C_s})$ and $(\mathbf{I_w})$. Then, *a.s.*

$$\lim_{\gamma\to 0}\ \lim_{N,n\to+\infty}\hat{\vartheta}_{N,n,\gamma} = \vartheta_0 \quad \text{and} \quad \lim_{N,n\to+\infty}\hat{\vartheta}_{N,n} = \vartheta_0.$$

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Remarks

- Proof: uniform convergence of the occupation measure to the marginal invariant distribution.
- Under $(\mathbf{C_w})$, we need to discretize Θ to keep the uniform convergence.

4 Rate of convergence

($\mathbf{I_s}$): There exists a constant $C > 0$ and a parameter $\varsigma \in (0,1]$ such that:

$$\forall \vartheta \in \Theta, \quad d(\nu_\vartheta, \nu_{\vartheta_0}) \geq C|\vartheta - \vartheta_0|^\varsigma.$$

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Let $\mu, \nu \in \mathcal{M}_1(\mathbb{R}^d)$. Let $d_{CF,p}$ and d_s be defined in the following way:

(*i*) let $g_p(\xi) := c_p(1 + |\xi|^2)^{-p}$ and $c_p := \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^{-p} d\xi\right)^{-1}$,

$$d_{CF,p}(\mu, \nu) := \left(\int_{\mathbb{R}^d} \left|\mu(e^{i\langle \xi, \cdot \rangle}) - \nu(e^{i\langle \xi, \cdot \rangle})\right|^2 g_p(\xi) d\xi\right)^{1/2}.$$

(*ii*) Let $\{f_i \ ; \ i \geq 1\}$ be a family of $\mathcal{C}_b^1$, supposed to be dense in the space $\mathcal{C}_b^0$.

$$d_s(\mu, \nu) := \sum_{i=0}^{+\infty} 2^{-i}(|\mu(f_i) - \nu(f_i)| \wedge 1).$$

Quadratic error (for $d_{CF,p}$ and d_s)

Theorem (Panloup, Tindel, V. '19)

Assume $(\mathbf{H_0})$, $(\mathbf{C_s})$ and $(\mathbf{I_s})$ for some given $\varsigma \in (0,1]$ hold true.

$$\mathbb{E}\left[\left|\hat{\vartheta}_{N,n,\gamma} - \vartheta_0\right|^2\right] \leq C_q \left(n^{-\frac{q}{2}(2-(2H\vee 1))} + \gamma^{qH} + (N\gamma)^{-\tilde{\eta}} \right)$$

with $q = 2/\varsigma$ and $\tilde{\eta} := \frac{q^2}{2(q+d)}(2-(2H\vee 1))$.

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$$\mathbb{E}\left[|\hat{\vartheta}_{N,n,\gamma} - \vartheta_0|^2\right]$$

$$(\mathbf{I_s}) \qquad \Rightarrow \mathbb{E}\left[d\left(\nu_{\hat{\vartheta}_{N,n,\gamma}}, \nu_{\vartheta_0}\right)^q\right]$$

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Triangle inequality + def of $\hat{\vartheta}$ $$\Rightarrow d\left(\nu_{\vartheta_0}, \frac{1}{n}\sum_{k=0}^{n-1} \delta_{Y_{t_k}}\right), \quad d\left(\nu_{\vartheta}, \frac{1}{T}\int_0^T \delta_{Y_t^{\vartheta}}\, dt\right)$$

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$\Rightarrow$ **Concentration inequalities.**

Let $Y := (Y_t)_{t \geqslant 0}$ be an $\mathbb{R}^d$-valued process such that

$$Y_t = x + \int_0^t b(Y_s)\mathrm{d}s + \sigma B_t.$$

where $x \in \mathbb{R}^d$, $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma \in \mathcal{M}_d(\mathbb{R})$ and B is a d-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

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Aim : For all function $f : \mathbb{R}^d \to \mathbb{R}$ Lipschitz, we wish to control :

* $\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n}(f(Y_{k\Delta}) - \mathbb{E}[f(Y_{k\Delta})]) > r\right)$ for a fixed $\Delta > 0$,
* $\mathbb{P}\left(\frac{1}{T}\int_0^T (f(Y_t) - \mathbb{E}[f(Y_t)])\mathrm{d}t > r\right)$

with respect to n and T.

5. Concentration inequalities for fractional SDEs
   - Transportation inequalities
   - Main results
   - Sketch of proof

Let (E, d) be a metric space. Given $p \geqslant 1$ and two probability measures μ and ν on E, the Wasserstein distance is defined by

$$\mathcal{W}_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\int_{E \times E} d(x, y)^p \mathrm{d}\pi(x, y) \right)^{1/p},$$

where $\Pi(\mu, \nu) := \{\pi \in \mathcal{M}_1(E \times E)$ telles que $\pi(., E) = \mu(.)$ et $\pi(E, .) = \nu(.)\}$.

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where $\Pi(\mu,\nu) := \{\pi \in \mathcal{M}_1(E\times E) \text{ telles que } \pi(.,E) = \mu(.) \text{ et } \pi(E,.) = \nu(.)\}$.
The relative entropy of ν with respect to μ is defined by

$$\mathbf{H}(\nu|\mu) = \begin{cases} \int \log\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) \mathrm{d}\nu, & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

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Then, we say that μ satisfies an L^p-transportation inequality with constant $C \geqslant 0$ (denoted by $\mu \in T_p(C)$) if for any probability measure ν,

$$\mathcal{W}_p(\mu,\nu) \leqslant \sqrt{2C\mathbf{H}(\nu|\mu)}.$$

Theorem (Bobkov and Götze '99)

Let (E, d) be a metric space and μ a probability measure on E. Then, $\mu \in T_1(C)$ if and only if for any μ-integrable Lipschitz function $F : (E, d) \to \mathbb{R}$ we have for all $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left[e^{\lambda(F(X)-\mathbb{E}[F(X)])}\right] \leqslant \exp\left(\frac{\lambda^2}{2} C \|F\|_{\mathrm{Lip}}^2\right)$$

with $\mathcal{L}(X) = \mu$. In that case,

$$\mathbb{P}\left(F(X) - \mathbb{E}[F(X)] > r\right) \leqslant \exp\left(-\frac{r^2}{2C\|F\|_{\mathrm{Lip}}^2}\right), \quad \forall r > 0.$$

Results for our fractional SDE : $\mu = \mathcal{L}((Y_t)_{t\in[0,T]})$

- For $H > 1/2$, Saussereau shows that $\mu \in T_2(C_T)$ for two metrics on $\mathcal{C}([0,T],\mathbb{R}^d)$:

$$d_2(\gamma_1,\gamma_2) = \left(\int_0^T |\gamma_1(t) - \gamma_2(t)|^2 dt\right)^{1/2} \quad \text{and} \quad d_\infty(\gamma_1,\gamma_2) = \sup_{t\in[0,T]} |\gamma_1(t) - \gamma_2(t)|.$$

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- For $H \in (0,1)$ (and even for general Gaussian process), Riedel shows that $\mu \in T_2(C_T)$ for the metric d_∞.

Theorem (Saussereau '12)

Let $H > 1/2$. There exists $C > 0$ such that for all Lipschitz function $f : \left(\mathbb{R}^d, |\cdot|\right) \to (\mathbb{R}, |\cdot|)$ and for all $r \geqslant 0$,

$$\mathbb{P}\left(\frac{1}{T}\int_0^T (f(Y_t) - \mathbb{E}[f(Y_t)])\mathrm{d}t > r\right) \leqslant \exp\left(-\frac{r^2 T^{2-2H}}{4C\|f\|_{\mathrm{Lip}}^2}\right).$$

5 Concentration inequalities for fractional SDEs

- Transportation inequalities
- Main results
- Sketch of proof

Hypothesis : We assume that there exist $\alpha, L > 0$ such that: for all $x, y \in \mathbb{R}^d$,

$$\langle b(x) - b(y), x - y \rangle \leq -\alpha |x-y|^2 \quad \text{et} \quad |b(x) - b(y)| \leq L|x-y|.$$

Theorem (V. '19)

Let $H \in (0,1)$ and $\Delta > 0$. Let $n \in \mathbb{N}^*$ and $T \geqslant 1$. Then,

(i) $\mathcal{L}((Y_{k\Delta})_{1\leq k \leq n}) \in T_1(2C_{H,\Delta} n^{2H\vee 1})$ for the metric $d_n(x,y) := \sum_{k=1}^n |x_k - y_k|$,

(ii) $\mathcal{L}((Y_t)_{t\in[0,T]}) \in T_1(2\tilde{C}_H T^{2H\vee 1})$ for the metric $d_T(x,y) := \int_0^T |x_t - y_t| dt$.

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Consequently,

(*i*) for all Lipschitz function $f : (\mathbb{R}^d, |\cdot|) \to (\mathbb{R}, |\cdot|)$ and for all $r \geqslant 0$,

$$\mathbb{P}\left(\frac{1}{n} \sum_{k=1}^{n} (f(Y_{k\Delta}) - E[f(Y_{k\Delta})]) > r \right) \leqslant \exp\left(- \frac{r^2 n^{2-(2H \vee 1)}}{4C_{H,\Delta} \|f\|_{\mathrm{Lip}}^2} \right).$$

(*ii*) for all Lipschitz function $f : (\mathbb{R}^d, |\cdot|) \to (\mathbb{R}, |\cdot|)$ and for all $r \geqslant 0$,

$$\mathbb{P}\left(\frac{1}{T} \int_0^T (f(Y_t) - \mathbb{E}[f(Y_t)]) \mathrm{d}t > r \right) \leqslant \exp\left(- \frac{r^2 T^{2-(2H \vee 1)}}{4\tilde{C}_H \|f\|_{\mathrm{Lip}}^2} \right).$$

We set

$$F_Y := \frac{1}{n}\sum_{k=1}^{n} f(Y_{k\Delta}).$$

Assume $\Delta = 1$ for the sake of simplicity. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration associated to the underlying standard Brownian motion W (see the Voltera representation). We set $M_k = \mathbb{E}[F_Y|\mathcal{F}_k]$.
Then

$$F_Y - \mathbb{E}[F_Y] = M_n - M_0 = \sum_{k=1}^{n} M_k - M_{k-1}.$$

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We show the existence of a deterministic sequence (u_k) such that

$$\mathbb{E}\left[e^{\lambda(M_k - M_{k-1})} \middle| \mathcal{F}_{k-1}\right] \leq e^{\lambda^2 u_k}$$

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$$\mathbb{E}\left[e^{\lambda(M_k - M_{k-1})}\Big| \mathcal{F}_{k-1}\right] \leq e^{\lambda^2 u_k} \Longrightarrow \quad \mathbb{E}\left[e^{\lambda(M_n - M_0)}\right] \leq e^{\lambda^2 \sum_{k=1}^{n} u_k}$$

Since $\mathbb{E}[M_k - M_{k-1}|\mathcal{F}_{k-1}] = 0$, we are thus reduced to estimate

$$\mathbb{E}[|M_k - M_{k-1}|^p|\mathcal{F}_{k-1}] \quad ,\forall p \geq 2.$$

Recall that :

$$Y_t = x + \int_0^t b(Y_s)\mathrm{d}s + \sigma \int_0^t K_H(t,s)\mathrm{d}W_s.$$

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We can see Y_t as a functional $\Phi : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}^d$ depending on the time, the initial condition x and the Brownian motion :

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Let $k \geq 1$,

$$\begin{aligned}
&|M_k - M_{k-1}| \\
&= |\mathbb{E}[F_Y|\mathcal{F}_k] - \mathbb{E}[F_Y|\mathcal{F}_{k-1}]| = \left| \frac{1}{n} \sum_{t=k}^{n} \mathbb{E}[f(Y_t)|\mathcal{F}_k] - \mathbb{E}[f(Y_t)|\mathcal{F}_{k-1}] \right| \\
&\leqslant \frac{\|f\|_{\mathrm{Lip}}}{n} \int_\Omega \sum_{t=k}^{n} \left| \Phi_t\left(x, W_{[0,k]} \sqcup \tilde{w}_{[k,t]}\right) - \Phi_t\left(x, W_{[0,k-1]} \sqcup \tilde{w}_{[k-1,t]}\right) \right| \mathbb{P}_W(\mathrm{d}\tilde{w}) \\
&\leqslant \frac{\|f\|_{\mathrm{Lip}}}{n} \int_\Omega \sum_{u=1}^{n-k+1} \left| X_u - \tilde{X}_u \right| \mathbb{P}_W(\mathrm{d}\tilde{w})
\end{aligned}$$

Using the SDE, we get for all $u \geq 1$,

$$X_u - \tilde{X}_u = \int_0^u b(X_s) - b(\tilde{X}_s)\mathrm{d}s + \sigma \int_{k-1}^{k} K_H(u+k-1, s)\mathrm{d}(W - \tilde{w})_s$$

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Key elements in the proof : Assumption on the drift b + precise estimates on the kernel K_H.

6 Discussion
- Identifiability assumption (dimension 1)
- Numerical discussion

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Lemma

We call μ_ϑ the marginal invariant distribution of X^ϑ defined by (7.1). Then for all $\vartheta_1, \vartheta_2 \in [m, M]$, we have

$$d_{CF,p}(\mu_{\vartheta_1}, \mu_{\vartheta_2}) \geq c_{m,M,H} \left|\vartheta_1 - \vartheta_2\right|.$$

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$$dX_t^{\vartheta} = -\vartheta X_t^{\vartheta} dt + \sigma dB_t \tag{7.1}$$

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In this case, we know (see Buchmann, Klüppelberg (2005)) that

$$\mu_\vartheta = \mathcal{N}(0, \sigma_\vartheta^2) \quad \text{with} \quad \sigma_\vartheta^2 = \frac{c_H}{\vartheta^{2H}}.$$

Small perturbation of fOU

$$dY_t^{\lambda,\vartheta} = \left[-\vartheta Y_t^{\lambda,\vartheta} + \lambda b_\vartheta(Y_t^{\lambda,\vartheta})\right] dt + \sigma dB_t \tag{7.2}$$

where $\vartheta \in [m, M]$ with $0 < m < M < +\infty$, and $\lambda \leq \lambda_0(m, M)$ with $\lambda_0(m, M)$ small enough.

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Proposition

Let $Y^{\lambda,\vartheta}$ be the process defined by (7.2). Assume (without loss of generality) that b_ϑ and its derivatives are all bounded by 1. Then for any $\vartheta_1, \vartheta_2 \in [m, M]$:

$$d_{CF,p}(\nu_{\vartheta_1}, \nu_{\vartheta_2}) \geq c_{m,M,H}|\vartheta_1 - \vartheta_2|. \tag{7.3}$$

6 Discussion

- Identifiability assumption (dimension 1)
- Numerical discussion

- Observations $(Y_{t_k})_{1 \le k \le n}$:
  - Euler-scheme with step $\underline{\gamma}$ with parameter ϑ_0.
  - Selection of a subsequence of observations : $t_k = k\gamma$ with $\gamma = k_0\underline{\gamma}$.

$$\vartheta_0 = 2, \quad \underline{\gamma} = 10^{-3}, \quad \gamma = 10^{-2}, \quad n = 3 \times 10^4.$$

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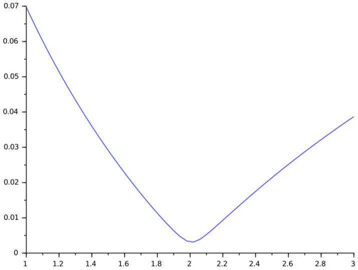
- Distance $d_{CF,2}$: approximation of integral by a sum.
- Wasserstein distance of order $p\in\{1,2,4\}$: in dimension 1, we have

$$\mathcal{W}_p\left(\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}, \frac{1}{n}\sum_{i=1}^{n}\delta_{y_i}\right) = \left(\frac{1}{n}\sum_{i=1}^{n}|x_{(i)} - y_{(i)}|^p\right)^{1/p}$$

where $x_{(1)}\leq x_{(2)}\leq\cdots\leq x_{(n)}$ and $y_{(1)}\leq y_{(2)}\leq\cdots\leq y_{(n)}$.

Fractional Ornstein Ulhenbeck process



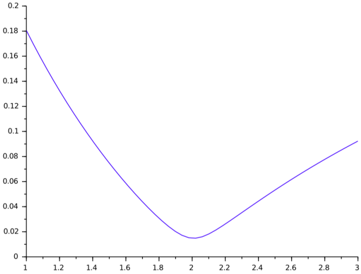


Figure: $\vartheta \mapsto \mathcal{F}_{d_{CF,2}}(\vartheta)$ for $H = 0.3$ (left) and $H = 0.7$ (right).

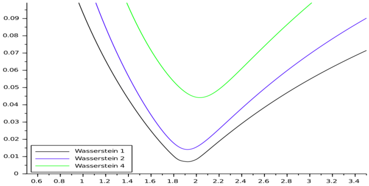


Figure: $\vartheta \mapsto \mathcal{F}_{\mathcal{W}_p}(\vartheta)$, $H = 0.3$.

Non linear drift : $b_\vartheta(x) = -x(1 + \cos(\vartheta x))$

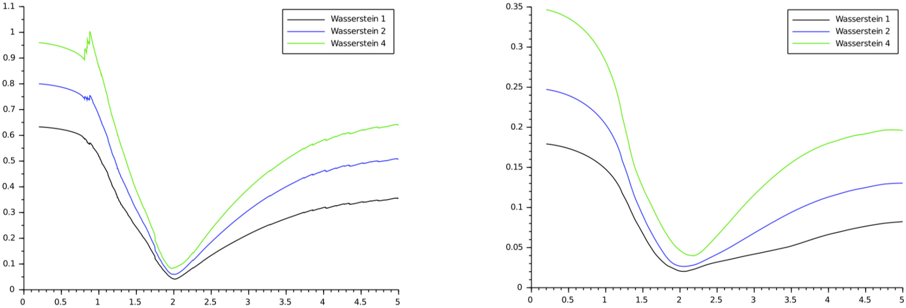


Figure: $\vartheta \mapsto \mathcal{F}_{\mathcal{W}_p}(\vartheta)$ for $H = 0.3$ (left) and $H = 0.7$ (right).

Thank you !

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