A drift estimation procedure for stochastic differential equations with additive fractional noise

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Joint work with Fabien Panloup & Samy Tindel.

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Outline

1. Introduction
   - Fractional Brownian motion
   - Ergodicity of fractional SDEs and approximation of stationary regime
   - Overview on drift estimation for fractional diffusion.

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Definition

Let $H \in (0,1)$. The d-dimensional *fractional Brownian motion* (fBm) with Hurst parameter H, denoted by $(B_t)_{t\geq 0}$, is a centered Gaussian process with covariance function given by :

$$\mathbb{E}[B_t^i B_s^j] = \frac{1}{2}\delta_{ij}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right] \qquad \text{for all } t, s \geq 0.$$

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- Stationary increments :

$$\mathbb{E}\left[(B_t^i - B_s^i)(B_t^j - B_s^j)\right] = \delta_{ij}|t-s|^{2H}.$$

- Self-similarity :

$$\mathcal{L}((B_{ct})_{t\geq 0}) = \mathcal{L}(c^H (B_t)_{t\geq 0}) \quad \text{for all } c > 0.$$

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Remarks

- ▷ The fBm is neither a semimartingale nor a Markov process except for $H = 1/2$. In that case, B is the standard Brownian motion and has independent increments.
- ▷ Regularity: a.s. locally Hölder for all $\beta < H$.

Let $(W_t)_{t\in\mathbb{R}}$ be a standard Brownian motion.

Proposition (Mandelbrot Van Ness representation)

$$B_t := \int_{\mathbb{R}} (t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \mathrm{d}W_s, \quad t \in \mathbb{R},$$

where $x_+ = \max(0, x)$.

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SDE with additive fractional noise:

$$dY_t = b(Y_t)dt + \sigma dB_t. \qquad \textbf{(E)}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma \in \mathbb{M}_{d\times d}$

Hairer (2005):

- Homogeneous Markovian structure: $(Y_t, (W_s)_{s\leq t})_{t\geq 0}$ (state space: $\mathbb{R}^d \times \mathcal{W}$).
- Existence of invariant distribution: $\mu_\star \in \mathcal{M}_1(\mathbb{R}^d \times \mathcal{W})$.
- Uniqueness of $\mu_\star$ and rate of convergence in total variation distance: $t^{-\alpha_H}$ with

$$\alpha_H = \begin{cases} H(1-2H) & \text{if } H \in (0, 1/4] \\ 1/8 & \text{if } H \in (1/4, 1) \setminus \left\{\frac{1}{2}\right\} \end{cases} .$$

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Multiplicative case:

- Fontbona, Panloup (2017): $H \in (1/2, 1)$.
- Deya, Panloup and Tindel (2019): $H \in (1/3, 1/2)$.

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Euler scheme for a fixed step $\gamma > 0$: $Z_0 = Y_0$,

$$Z_{(n+1)\gamma} = Z_{n\gamma} + \gamma b(Z_{n\gamma}) + \sigma(B_{(n+1)\gamma} - B_{n\gamma}). \qquad (\mathbf{E}_\gamma)$$

Theorem (Cohen, Panloup '11) (Cohen, Panloup, Tindel '14)

$$\lim_{\gamma \to 0} \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{Z_{k\gamma}} = \bar{\mu}_\star \quad a.s.$$

in the sense of weak convergence on $\mathcal{M}_1(\mathbb{R}^d)$.

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1. On fractional Ornstein-Ulhenbeck processes (fOU) : $b_\vartheta(x) = -\vartheta x$.
    - Le Breton (1998)
    - Klepstyna, Le Breton (2002)
    - Hu, Nualart (2010)
    - Belfadi, Es-Sebaiy, Ouknine (2011)

2. On linear drift : $b_\vartheta(x) = \vartheta b(x)$
    - Tudor, Viens (2007)

3. Non parametric estimation in dimension 1
    - Comte, Marie (2018)

$\Longrightarrow$ *Most of them : continuous-time observation of the process.*

4. Discrete-time observations : Neuenkirsh, Tindel (2014)
    - $b_\vartheta(x) = \nabla F(x, \vartheta)$.
    - $\bar{Y}_0$ stationary solution : $\mathbb{E}\left[|b_{\vartheta_0}(\bar{Y}_0)|^2\right] = \mathbb{E}\left[|b_\vartheta(\bar{Y}_0)|^2\right] \quad \Leftrightarrow \quad \vartheta = \vartheta_0$.

2. Model and contruction of the estimators

Let Y be an $\mathbb{R}^d$-valued process such that $Y_0 = y_0$ and

$$dY_t = b_{\vartheta_0}(Y_t)\,dt + \sigma dB_t \qquad (\mathbf{E}_{\vartheta_0})$$

where $\sigma \in \mathbb{M}_{d\times d}$ is an invertible matrix, $\vartheta_0 \in \Theta$ is the unknown parameter and $\{b_\vartheta(.) \mid \vartheta \in \Theta\}$ is a known family of functions.

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$(\mathbf{C_w})$: We have $b \in \mathcal{C}^{1,1}(\mathbb{R}^d \times \Theta; \mathbb{R}^d)$ and:

(i) $\forall x, y \in \mathbb{R}^d$, $\forall \vartheta \in \Theta$,

$$\langle b_\vartheta(x) - b_\vartheta(y),\, x - y\rangle \leq \beta - \alpha|x-y|^2 \quad \text{and} \quad |b_\vartheta(x) - b_\vartheta(y)| \leq L|x-y|$$

(ii) $\forall x \in \mathbb{R}^d$, $\forall \vartheta \in \Theta$,

$$|\partial_\vartheta b_\vartheta(x)| \leq C\left(1 + |x|^r\right).$$

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Observations : $\{Y_{t_k} \mid 0 \leq k < n\}$ and $t_{k+1} - t_k = \kappa > 0$.

$$
\begin{aligned}
(\mathbf{C_w}) \quad & \Rightarrow (\mathbf{E}_\vartheta) \text{ admits a unique invariant distribution} \\
& \Rightarrow \nu_\vartheta \in \mathcal{M}_1(\mathbb{R}^d) \text{ (marginal invariant distribution).}
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Identifiability assumptions: denote by d a distance on $\mathcal{M}_1(\mathbb{R}^d)$.

($\mathbf{I_w}$): We have: $\quad d(\nu_\vartheta, \nu_{\vartheta_0}) = 0 \quad \Leftrightarrow \quad \vartheta = \vartheta_0.$

($\mathbf{I_s}$): There exists a constant $C > 0$ and a parameter $\varsigma \in (0,1]$ such that:

$$\forall \vartheta \in \Theta, \quad d(\nu_\vartheta, \nu_{\vartheta_0}) \geq C|\vartheta - \vartheta_0|^\varsigma.$$

Approximation of ν_{ϑ_0}: $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{Y_{t_k}} \Rightarrow \nu_{\vartheta_0}$.

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Approximation of ν_{ϑ}: Euler scheme associated to $(\mathbf{E}_\vartheta)$: $Z_0^\vartheta = y_0$

$$\forall k \geq 0, \quad Z_{s_{k+1}}^\vartheta = Z_{s_k}^\vartheta + (s_{k+1} - s_k) b_\vartheta(Z_{s_k}^\vartheta) + \sigma\left(B_{s_{k+1}} - B_{s_k}\right).$$

where $s_0 = 0$ and (s_k) is an increasing sequence such that $\lim_{k\to+\infty} s_k = +\infty$.

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- Constant step Euler scheme $Z^{\vartheta,\gamma}$: $s_k = k\gamma$.
- Decreasing step Euler scheme Z^{ϑ}: $\gamma_k := s_k - s_{k-1}$ is a decreasing sequence and $\lim\limits_{k\to+\infty} \gamma_k = 0$.

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Estimators:

$$\hat{\vartheta}_{N,n,\gamma} = \operatorname*{argmin}_{\vartheta\in\Theta} d\left(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{Y_{t_k}},\ \frac{1}{N}\sum_{k=0}^{N-1}\delta_{Z^{\vartheta,\gamma}_{k\gamma}}\right),$$

$$\hat{\vartheta}_{N,n} = \operatorname*{argmin}_{\vartheta\in\Theta} d\left(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{Y_{t_k}},\ \frac{1}{s_N}\sum_{k=0}^{N-1}\gamma_{k+1}\delta_{Z^{\vartheta}_{s_k}}\right).$$

where $d \in \mathcal{D}_p := \{\text{distances } d \text{ on } \mathcal{M}_1(\mathbb{R}^d);\ \exists\, c > 0, \forall \nu,\mu,\ d(\nu,\mu) \leq c\mathcal{W}_p(\nu,\mu)\}$

3. Consistency results

$$\hat{\vartheta}_{N,n,\gamma} = \underset{\vartheta\in\Theta}{\operatorname{argmin}}\ d\left(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{Y_{t_k}},\ \frac{1}{N}\sum_{k=0}^{N-1}\delta_{Z^{\vartheta,\gamma}_{k\gamma}}\right),$$

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Theorem (Panloup, Tindel, V. '19)

Assume $(\mathbf{H_0})$, $(\mathbf{C_s})$ and $(\mathbf{I_w})$. Then, *a.s.*

$$\lim_{\gamma\to 0}\ \lim_{N,n\to+\infty}\hat{\vartheta}_{N,n,\gamma} = \vartheta_0 \quad \text{and} \quad \lim_{N,n\to+\infty}\hat{\vartheta}_{N,n} = \vartheta_0.$$

Proposition

Let Θ be a compact set and $(\vartheta \mapsto L_r(\vartheta))_r$ denote a family of non-negative random functions. Assume that:

1. $\lim_r L_r(\vartheta) = L(\vartheta)$ a.s. and uniformly in ϑ.
2. $\vartheta \mapsto L(\vartheta)$ is non-random and continuous on Θ with a unique minimum in ϑ_0.
3. For any r, the set $\operatorname{argmin}\{L_r(\vartheta), \vartheta \in \Theta\}$ is nonempty.

For a fixed r, let $\hat{\vartheta}_r \in \operatorname{argmin}\{L_r(\vartheta), \vartheta \in \Theta\}$, then

$$\lim_r \hat{\vartheta}_r = \vartheta_0.$$

Remarks

- Here, $L(\vartheta) = d(\nu_\vartheta, \nu_{\vartheta_0})$.
- Proof: uniform convergence of the occupation measure to the marginal invariant distribution.
- Under $(\mathbf{C_w})$, we need to discretize Θ to keep the uniform convergence.

4. Rate of convergence
   - Bound on the quadratic error
   - Identifiability assumption (dimension 1)

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Let $\mu, \nu \in \mathcal{M}_1(\mathbb{R}^d)$. Let $d_{CF,p}$ and d_s be defined in the following way:

(i) let $g_p(\xi) := c_p(1 + |\xi|^2)^{-p}$ and $c_p := \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^{-p} d\xi\right)^{-1}$,

$$d_{CF,p}(\mu, \nu) := \left(\int_{\mathbb{R}^d} \left|\mu(e^{i\langle \xi, \cdot \rangle}) - \nu(e^{i\langle \xi, \cdot \rangle})\right|^2 g_p(\xi) d\xi\right)^{1/2}.$$

(ii) Let $\{f_i \ ; \ i \geq 1\}$ be a family of $\mathcal{C}_b^1$, supposed to be dense in the space $\mathcal{C}_b^0$.

$$d_s(\mu, \nu) := \sum_{i=0}^{+\infty} 2^{-i} (|\mu(f_i) - \nu(f_i)| \wedge 1).$$

Quadratic error (for $d_{CF,p}$ and d_s)

Theorem (Panloup, Tindel, V. '19)

Assume $(\mathbf{H_0})$, $(\mathbf{C_s})$ and $(\mathbf{I_s})$ for some given $\varsigma \in (0,1]$ hold true.

$$\mathbb{E}\left[|\hat{\vartheta}_{N,n,\gamma} - \vartheta_0|^2\right] \leq C_q \left(n^{-\frac{q}{2}(2-(2H\vee 1))} + \gamma^{qH} + (N\gamma)^{-\tilde{\eta}}\right)$$

with $q = 2/\varsigma$ and $\tilde{\eta} := \frac{q^2}{2(q+d)}(2-(2H\vee 1))$.

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$$\begin{aligned} & \mathbb{E}\left[|\hat{\vartheta}_{N,n,\gamma} - \vartheta_0|^2\right] \\ (\mathbf{I_s}) \qquad & \Rightarrow \mathbb{E}\left[d\left(\nu_{\hat{\vartheta}_{N,n,\gamma}}, \nu_{\vartheta_0}\right)^q \right] \end{aligned}$$

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$(\mathbf{I_s})$ $$\Rightarrow \mathbb{E}\left[d\left(\nu_{\hat{\vartheta}_{N,n,\gamma}}, \nu_{\vartheta_0}\right)^q\right]$$

Triangle inequality + def of $\hat{\vartheta}$ $$\Rightarrow d\left(\nu_{\vartheta_0}, \frac{1}{n}\sum_{k=0}^{n-1}\delta_{Y_{t_k}}\right), \quad d\left(\nu_{\vartheta}, \frac{1}{T}\int_0^T \delta_{Y_t^{\vartheta}}\,dt\right)$$

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$\Rightarrow$ **Concentration inequalities** (see Varvenne (2019)).

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Case of fOU

$$dX_t^{\vartheta} = -\vartheta X_t^{\vartheta} dt + \sigma dB_t \tag{5.1}$$

with $\vartheta \in [m, M]$ and $0 < m < M < +\infty$.

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Lemma

We call μ_ϑ the marginal invariant distribution of X^ϑ defined by (5.1). Then for all $\vartheta_1, \vartheta_2 \in [m, M]$, we have

$$d_{CF,p}(\mu_{\vartheta_1}, \mu_{\vartheta_2}) \geq c_{m,M,H} \, |\vartheta_1 - \vartheta_2|.$$

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In this case, we know (see Buchmann, Klüppelberg (2005)) that

$$\mu_\vartheta = \mathcal{N}(0, \sigma_\vartheta^2) \quad \text{with} \quad \sigma_\vartheta^2 = \frac{c_H}{\vartheta^{2H}}.$$

Small perturbation of fOU

$$dY_t^{\lambda,\vartheta} = \left[-\vartheta Y_t^{\lambda,\vartheta} + \lambda b_\vartheta(Y_t^{\lambda,\vartheta})\right] dt + \sigma dB_t \tag{5.2}$$

where $\vartheta \in [m, M]$ with $0 < m < M < +\infty$, and $\lambda \leq \lambda_0(m, M)$ with $\lambda_0(m, M)$ small enough.

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Proposition

Let $Y^{\lambda,\vartheta}$ be the process defined by (5.2). Assume (without loss of generality) that b_ϑ and its derivatives are all bounded by 1. Then for any $\vartheta_1, \vartheta_2 \in [m, M]$:

$$d_{CF,p}(\nu_{\vartheta_1}, \nu_{\vartheta_2}) \geq c_{m,M,H}|\vartheta_1 - \vartheta_2|. \tag{5.3}$$

5 Numerical discussion

- Observations $(Y_{t_k})_{1\leq k\leq n}$:
  - Euler-scheme with step $\underline{\gamma}$ with parameter ϑ_0.
  - Selection of a subsequence of observations : $t_k = k\gamma$ with $\gamma = k_0\underline{\gamma}$.

$$\vartheta_0 = 2, \quad \underline{\gamma} = 10^{-3}, \quad \gamma = 10^{-2}, \quad n = 3\times 10^4.$$

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$$\vartheta_0 = 2, \quad \underline{\gamma} = 10^{-3}, \quad \gamma = 10^{-2}, \quad n = 3\times 10^4.$$

- Euler Scheme $(Z^{\vartheta}_{k\gamma})_{1\leq k\leq N}$ for different values of ϑ :

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- Distance $d_{CF,2}$: approximation of integral by a sum.
- Wasserstein distance of order $p \in \{1, 2, 4\}$: in dimension 1, we have

$$\mathcal{W}_p \left(\frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}, \frac{1}{n} \sum_{i=1}^{n} \delta_{y_i} \right) = \left(\frac{1}{n} \sum_{i=1}^{n} |x_{(i)} - y_{(i)}|^p \right)^{1/p}$$

where $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$ and $y_{(1)} \le y_{(2)} \le \cdots \le y_{(n)}$.

Fractional Ornstein Ulhenbeck process

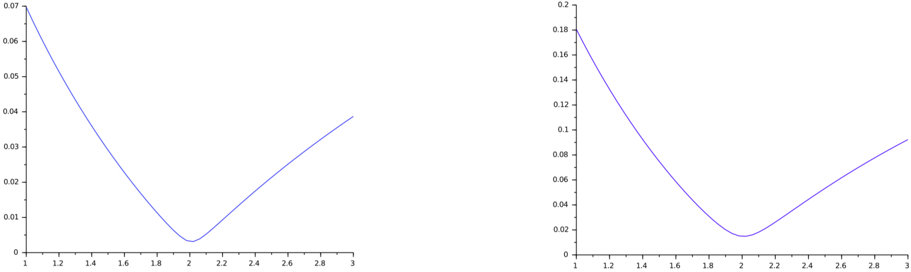

Figure: $\vartheta \mapsto \mathcal{F}_{d_{CF,2}}(\vartheta)$ for $H = 0.3$ (left) and $H = 0.7$ (right).

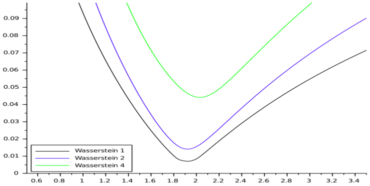

Figure: $\vartheta \mapsto \mathcal{F}_{\mathcal{W}_p}(\vartheta)$, $H = 0.3$.

Non linear drift : $b_\vartheta(x) = -x(1+\cos(\vartheta x))$

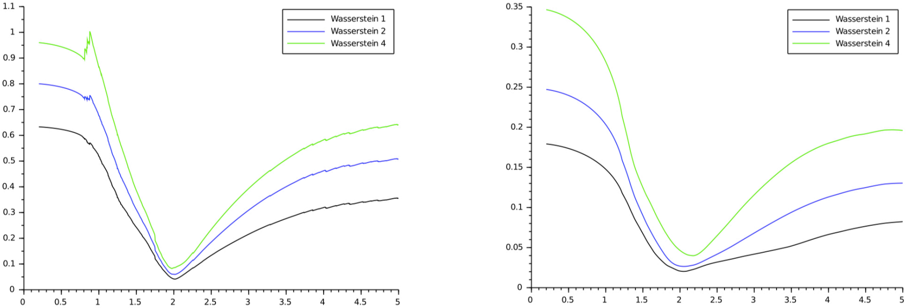

Figure: $\vartheta \mapsto \mathcal{F}_{\mathcal{W}_p}(\vartheta)$ for $H = 0.3$ (left) and $H = 0.7$ (right).

Thank you !

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