Rate of convergence to equilibrium for discrete-time stochastic dynamics with memory

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Setting

Let $X := (X_n)_{n \geqslant 0}$ be an $\mathbb{R}^d$-valued process such that

$$X_{n+1} = F(X_n, \Delta_{n+1})$$

where $(\Delta_n)_{n \in \mathbb{Z}}$ is an ergodic stationary Gaussian sequence with d-independent components and $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is (at least) continuous.

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Questions:

- Definition of invariant distribution in this a priori non-Markovian setting ?
- Existence and uniqueness of such measure ? Rate of convergence to equilibrium ?

Example : Euler scheme of a Gaussian SDE

Let $h > 0$ be fixed.

$$X_{n+1} = X_n + hb(X_n) + \sigma(X_n)\Delta_{n+1}$$

with $\Delta_{n+1} := Z_{(n+1)h} - Z_{nh}$ where (Z_t) is a Gaussian process with stationary increments.
Then,

$$X_{n+1} = F_h(X_n, \Delta_{n+1})$$

and

$$\begin{aligned} F_h : \quad & \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \\ & (x, w) \mapsto x + hb(x) + \sigma(x)w. \end{aligned}$$

Example : Euler scheme of a Gaussian SDE

Example of noise process (Z_t)

Fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$, denoted by $(B_t)_{t\in\mathbb{R}}$. The fBm is a centered Gaussian process with covariance function given by: for all $t, s \in \mathbb{R}$

$$\mathbb{E}[B_t^i B_s^j] = \frac{1}{2}\delta_{ij}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right], \quad i,j \in \{1,\dots,d\}.$$

In particular, the fBm increments are stationary: for all $t, s \in \mathbb{R}$

$$\mathbb{E}[(B_t^i - B_s^i)(B_t^j - B_s^j)] = \delta_{ij}|t-s|^{2H}, \quad i,j \in \{1,\dots,d\}.$$

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Remark: The fBm is neither a semimartingale nor a Markov process, except for $H = 1/2$. In this case, B is the standard Brownian motion and has independent increments.

Let $\mathcal{X} := \mathbb{R}^d$ be the state space and $\mathcal{W} := (\mathbb{R}^d)^{\mathbb{Z}^-}$ be the noise space.
Idea:

$$(X_n)_{n\in\mathbb{N}} \in \mathcal{X}^{\mathbb{N}} \dashrightarrow (X_n, (\Delta_{n+k})_{k\leqslant 0})_{n\in\mathbb{N}} \in (\mathcal{X} \times \mathcal{W})^{\mathbb{N}}$$

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Equivalent system:

$$(X_{n+1}, (\Delta_{n+1+k})_{k\leqslant 0}) = \varphi\left((X_n, (\Delta_{n+k})_{k\leqslant 0}), \Delta_{n+1}\right) \qquad (2.1)$$

where

$$\begin{aligned}\varphi : (\mathcal{X} \times \mathcal{W}) \times \mathbb{R}^d &\to \mathcal{X} \times \mathcal{W} \\ ((x, w), \delta) &\mapsto (F(x, \delta), w \sqcup \delta).\end{aligned}$$

Transition kernel: For all measurable function $g : \mathcal{X} \times \mathcal{W} \to \mathbb{R}$, $\mathcal{Q} : \mathcal{X} \times \mathcal{W} \to \mathcal{M}_1(\mathcal{X} \times \mathcal{W})$ is defined by :

$$\int_{\mathcal{X}\times\mathcal{W}} g(x', w')\mathcal{Q}((x, w), (\mathrm{d}x', \mathrm{d}w')) = \int_{\mathbb{R}^d} g(F(x, \delta), w \sqcup \delta)\mathcal{P}(w, \mathrm{d}\delta).$$

where $\mathcal{P}(w, \mathrm{d}\delta) := \mathcal{L}(\Delta_{n+1}|(\Delta_{n+k})_{k\leqslant 0} = w)$.

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Definition

A measure $\mu \in \mathcal{M}_1(\mathcal{X} \times \mathcal{W})$ is said to be an **invariant distribution** for our system if it is invariant by $\mathcal{Q}$, i.e.

$$\mathcal{Q}\mu = \mu.$$

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Uniqueness: Let $\mathcal{S} : \mathcal{M}_1(\mathcal{X} \times \mathcal{W}) \to \mathcal{M}_1(\mathcal{X}^{\mathbb{N}})$ be the application which maps μ into $S\mu := \mathcal{L}((X_n^\mu)_{n\geqslant 0})$. Then

$$\mu \simeq \nu \iff \mathcal{S}\mu = \mathcal{S}\nu \quad (\star)$$

Moving average representation

Wold's decomposition theorem,

$$\forall n \in \mathbb{Z}, \quad \Delta_n = \sum_{k=0}^{+\infty} a_k \xi_{n-k} \tag{3.1}$$

with

$$\begin{cases} (a_k)_{k \geqslant 0} \in \mathbb{R}^{\mathbb{N}} \ \text{ such that } \ a_0 \neq 0 \ \text{ and } \ \sum_{k=0}^{+\infty} a_k^2 < +\infty \\ (\xi_k)_{k \in \mathbb{Z}} \ \text{ an i.i.d sequence such that } \xi_1 \sim \mathcal{N}(0, I_d). \end{cases}$$

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Remarks

- Without loss of generality, we assume that $a_0 = 1$. If $a_0 \neq 1$, we can come back to this case by setting $\tilde{\Delta}_n = \sum_{k=0}^{+\infty} \tilde{a}_k \xi_{n-k}$ with $\tilde{a}_k = a_k / a_0$.

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- The memory induced by the noise is quantified by $(a_k)_{k\geqslant 0}$.

Preliminary tool : a Toeplitz type operator

Definition

Let $\mathbf{T_a}$ be defined on $\ell_a(\mathbb{Z}^-, \mathbb{R}^d) := \left\{ w \in (\mathbb{R}^d)^{\mathbb{Z}^-} \;\middle|\; \forall k \geqslant 0, \left|\sum_{l=0}^{+\infty} a_l w_{-k-l}\right| < +\infty \right\}$ by

$$\forall w \in \ell_a(\mathbb{Z}^-, \mathbb{R}^d), \quad \mathbf{T_a}(w) = \left(\sum_{l=0}^{+\infty} a_l w_{-k-l} \right)_{k \geqslant 0}.$$

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Proposition

Let $\mathbf{T_b}$ be defined on $\ell_b(\mathbb{Z}^-, \mathbb{R}^d)$ with the following sequence $(b_k)_{k\geqslant 0}$

$$b_0 = \frac{1}{a_0} \quad \text{and} \quad \forall k \geqslant 1, \quad b_k = -\frac{1}{a_0} \sum_{l=1}^{k} a_l b_{k-l}.$$

Then, $\mathbf{T_b} = \mathbf{T_a}^{-1}$.

$(\mathbf{H_{poly}})$: The following conditions are satisfied,

- There exist $\rho, \beta > 0$ and $C_\rho, C_\beta > 0$ such that

$$\forall k \geqslant 0, \quad |a_k| \leqslant C_\rho (k+1)^{-\rho} \quad \text{and} \quad \forall k \geqslant 0, \quad |b_k| \leqslant C_\beta (k+1)^{-\beta}.$$

- There exist $\kappa \geqslant \rho + 1$ and $C_\kappa > 0$ such that

$$\forall k \geqslant 0, \quad |a_k - a_{k+1}| \leqslant C_\kappa (k+1)^{-\kappa}.$$

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Remark

▷ Even though $(a_k)_{k\geqslant 0}$ and $(b_k)_{k\geqslant 0}$ are intrinsically linked, there is no general rule which connects ρ to β.

Two general hypothesis on F.

Example : Euler scheme with step $h > 0$.

($\mathbf{H_{b,\sigma}}$): $b : \mathbb{R}^d \to \mathbb{R}^d$ is continuous, $\sigma : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ is bounded, continuous and $\sigma^{-1} : x \mapsto \sigma(x)^{-1}$ is well defined and continuous. Moreover,

- $\exists C > 0$ such that $\forall x \in \mathcal{X}, \quad |b(x)| \leqslant C(1 + |x|)$
- $\exists \tilde{\beta} \in \mathbb{R}$ and $\tilde{\alpha} > 0$ such that $\forall x \in \mathcal{X}, \quad \langle x, b(x) \rangle \leqslant \tilde{\beta} - \tilde{\alpha}|x|^2$.

Theorem

Assume the two hyposthesis on the function F. Then,

(i) *There exists an invariant distribution $\mu_\star$.*

(ii) *Assume that $(\mathbf{H_{poly}})$ is true with $\rho, \beta > 1/2$ and $\rho + \beta > 3/2$. Then, uniqueness holds for $\mu_\star$. Moreover, for all initial distribution μ_0 such that $\int_{\mathcal{X}} V(x)\Pi^*_{\mathcal{X}}\mu_0(dx) < +\infty$ and for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\|\mathcal{L}((X^{\mu_0}_{n+k})_{k\geqslant 0}) - \mathcal{S}\mu_\star\|_{TV} \leqslant C_\varepsilon \; n^{-(v(\beta,\rho)-\varepsilon)}.$$

where v is defined by

$$v(\beta,\rho) = \sup_{\alpha\in\left(\frac{1}{2}\vee\left(\frac{3}{2}-\beta\right),\rho\right)} \min\{1, 2(\rho-\alpha)\}(\min\{\alpha,\ \beta,\ \alpha+\beta-1\} - 1/2).$$

Example 1

When $(\Delta_n)_{n\in\mathbb{Z}} = (B_{nh} - B_{(n-1)h})_{n\in\mathbb{Z}}$ (with $h > 0$) we have

$$a_k^H \sim C_{h,H}(k+1)^{-(3/2-H)} \text{ and } |a_k^H - a_{k+1}^H| \leqslant C'_{h,H}(k+1)^{-(5/2-H)}.$$

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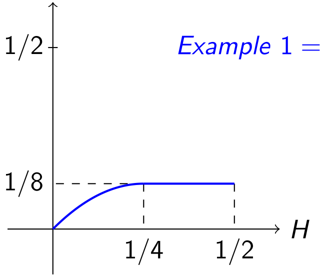
For $H \in (0, 1/2)$: $\forall k \geqslant 0, \quad |b_k^H| \leqslant C''_{h,H}(k+1)^{-(H+1/2)}$.

Rate of convergence



$$\textit{Example 1} = \begin{cases} H(1-2H) & \text{if} \quad H \in (0, 1/4] \\ 1/8 & \text{if} \quad H \in (1/4, 1/2) \end{cases}$$

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For $H \in (0,1/2)$: $\forall k \geqslant 0, \quad |b_k^H| \leqslant C''_{h,H}(k+1)^{-(H+1/2)}$.

Example 2

$$\text{If} \quad a_k = (k+1)^{-(3/2-H)}, \quad \text{then} \quad |b_k| \leqslant (k+1)^{-(3/2-H)}.$$

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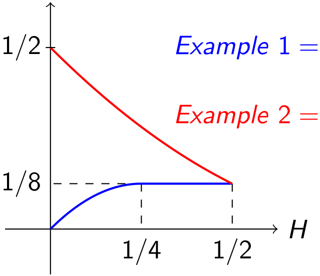

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$$\textit{Example 2} = (1-H)^2/2$$

References (continuous time setting) : Hairer (2005) - Fontbona & Panloup (2014) - Deya, Panloup & Tindel (2016)

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Scheme of coupling (discrete time setting) : We consider (X^1, X^2) the solution of the system :

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with initial conditions $(X^1_0, (\Delta^1_k)_{k\leqslant 0}) \sim \mu_0$ and $(X^2_0, (\Delta^2_k)_{k\leqslant 0}) \sim \mu_\star$.

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We have

$$\|\mathcal{L}((X^1_{n+k})_{k\geqslant 0}) - \mathcal{S}\mu_\star\|_{TV} \leqslant \mathbb{P}(\tau_\infty > n).$$

where $\tau_\infty := \inf\{n \geqslant 0 \mid X^1_k = X^2_k,\ \forall k \geqslant n\}$.

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We define the sequence of r.v. $(g_n)_{n\in\mathbb{Z}}$ by

$$\forall n \in \mathbb{Z}, \quad \xi^1_{n+1} = \xi^2_{n+1} + g_n, \qquad \text{hence} \qquad g_n = 0 \quad \forall n < 0.$$

Steps of the coupling procedure

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- **Step 2** : Try to keep the paths fastened together (specific to non-Markov process).
- **Step 3** : If Step 2 fails, impose $g_n = 0$ and wait long enough in order to allow Step 1 to be realized with a "controlled cost" and with a positive probability.

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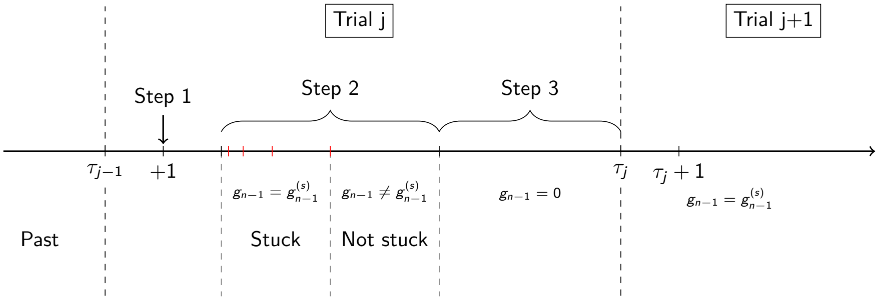
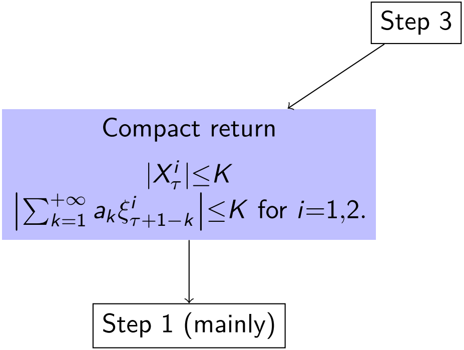
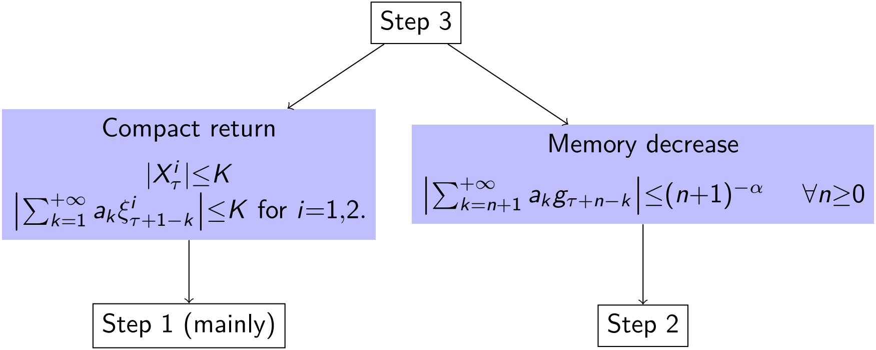


Step 3


Step 3
Compact return
$|X^i_\tau| \le K$
$\left|\sum_{k=1}^{+\infty} a_k \xi^i_{\tau+1-k}\right| \le K$ for $i=1,2$.
Step 1 (mainly)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Memory decrease
$\left|\sum_{k=n+1}^{+\infty} a_k g_{\tau+n-k}\right| \leq (n+1)^{-\alpha} \quad \forall n \geq 0$
Step 1 (mainly)
Step 2

Euler scheme (step 1)

At a given time $(\tau+1)$, we want to build $(\xi^1_{\tau+1}, \xi^2_{\tau+1})$ in order to get $X^1_{\tau+1} = X^2_{\tau+1}$, i.e.

$$X^1_\tau + hb(X^1_\tau) + \sigma(X^1_\tau) \sum_{k=0}^{+\infty} a_k \xi^1_{\tau+1-k} = X^2_\tau + hb(X^2_\tau) + \sigma(X^2_\tau) \sum_{k=0}^{+\infty} a_k \xi^2_{\tau+1-k}$$

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$$\iff \quad \xi^2_{\tau+1} = \Lambda_{\mathbf{X}}(\xi^1_{\tau+1}) \text{ where } \mathbf{X} = \left(X^1_\tau, X^2_\tau, \sum_{k=1}^{+\infty} a_k \xi^1_{\tau+1-k}, \sum_{k=1}^{+\infty} a_k \xi^2_{\tau+1-k} \right)$$

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Coupling Lemma to build $(\xi^1_{\tau+1}, \xi^2_{\tau+1})$ **such that**:

- $\xi^1_{\tau+1} \sim \mathcal{N}(0, I_d)$ and $\xi^2_{\tau+1} \sim \mathcal{N}(0, I_d)$,
- ensure $\mathbb{P}\left(\xi^2_{\tau+1} = \Lambda_{\mathbf{X}}(\xi^1_{\tau+1})\right) \geq \delta_K > 0$,
- $|\xi^1_{\tau+1} - \xi^2_{\tau+1}| \leqslant M_K$ a.s.

Euler scheme (step 2)

Keep the paths fastened : $X^1_{n+1} = X^2_{n+1} \ \ \forall n \geqslant \tau + 1$, i.e.

$$X_n^{1} + hb(X_n^{1}) + \sigma(X_n^{1}) \sum_{k=0}^{+\infty} a_k \xi^1_{n+1-k} = X_n^{1} + hb(X_n^{1}) + \sigma(X_n^{1}) \sum_{k=0}^{+\infty} a_k \xi^2_{n+1-k}$$

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Keep the paths fastened : $X^1_{n+1} = X^2_{n+1} \quad \forall n \geqslant \tau + 1$, i.e.

$$X^1_n + hb(X^1_n) + \sigma(X^1_n) \sum_{k=0}^{+\infty} a_k \xi^1_{n+1-k} = X^1_n + hb(X^1_n) + \sigma(X^1_n) \sum_{k=0}^{+\infty} a_k \xi^2_{n+1-k}$$

$$\iff \quad \forall n \geqslant \tau + 1, \quad \xi^1_{n+1} - \xi^2_{n+1} = g^{(s)}_n = -\sum_{k=1}^{+\infty} a_l g_{n-k}$$

$$\iff \quad \forall n \geqslant 1, \quad g^{(s)}_{\tau+n} = -\sum_{k=1}^{n} a_k g^{(s)}_{\tau+n-k} - \sum_{k=n+1}^{+\infty} a_k g_{\tau+n-k}. \tag{5.2}$$

Euler scheme (step 2)

Keep the paths fastened : $X^1_{n+1} = X^2_{n+1} \quad \forall n \geqslant \tau + 1$, i.e.

$$X_n^1 + hb(X_n^1) + \sigma(X_n^1)\sum_{k=0}^{+\infty} a_k \xi^1_{n+1-k} = X_n^1 + hb(X_n^1) + \sigma(X_n^1)\sum_{k=0}^{+\infty} a_k \xi^2_{n+1-k}$$

$$\iff \quad \forall n \geqslant \tau + 1, \quad \xi^1_{n+1} - \xi^2_{n+1} = g_n^{(s)} = -\sum_{k=1}^{+\infty} a_l g_{n-k}$$

$$\iff \quad \forall n \geqslant 1, \quad g^{(s)}_{\tau+n} = -\sum_{k=1}^{n} a_k g^{(s)}_{\tau+n-k} - \sum_{k=n+1}^{+\infty} a_k g_{\tau+n-k}. \qquad (5.2)$$

Coupling Lemma to build $((\xi^1_{\tau+n+1}, \xi^2_{\tau+n+1}))_{n\in[\![1,T]\!]}$ **such that**:

- ensure (5.2) with lower bounded positive probability,
- $\|(g_{\tau+n})_{n\in[\![1,T]\!]}\|$ a.s. upper bounded.

Aim : Determine for which value of $p > 0$ we can control $\mathbb{E}[\tau_\infty^p]$ since:

$$\mathbb{P}(\tau_\infty > n) \leqslant \frac{\mathbb{E}[\tau_\infty^p]}{n^p}$$

where $\tau_\infty := \inf\{n \geqslant 0 \mid X_k^1 = X_k^2,\ \forall k \geqslant n\}$.

Thank you !

($\mathbf{H_1}$): There exists $V : \mathbb{R}^d \to \mathbb{R}_+^*$ continuous such that $\lim_{|x| \to +\infty} V(x) = +\infty$ and $\exists \gamma \in (0,1)$ and $C > 0$ such that

$$\forall (x,w) \in \mathbb{R}^d \times \mathbb{R}^d, \quad V(F(x,w)) \leqslant \gamma V(x) + C(1 + |w|).$$

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$$\forall (x,w) \in \mathbb{R}^d \times \mathbb{R}^d, \quad V(F(x,w)) \leqslant \gamma V(x) + C(1 + |w|).$$

($\mathbf{H_2}$): Let $K > 0$. We assume that there exists $\tilde{K} > 0$ such that for every $\mathbf{X} := (x, x', y, y')$ in $B(0,K)^4$, there exist $\Lambda_{\mathbf{X}} : \mathbb{R}^d \to \mathbb{R}^d$, $M_K > 0$ and $C_{\tilde{K}}$ such that the following holds

- $\Lambda_{\mathbf{X}}$ is a bijection from $\mathbb{R}^d$ to $\mathbb{R}^d$. Moreover, it is a $\mathcal{C}^1$-diffeomorphism between two open sets U and D such that $\mathbb{R}^d \backslash U$ and $\mathbb{R}^d \backslash D$ are negligible sets.
- for all $u \in B(0, \tilde{K})$,

$$F(x, u + y) = F(x', \Lambda_{\mathbf{X}}(u) + y') \tag{5.3}$$
$$\text{and} \quad |\det(J_{\Lambda_{\mathbf{X}}}(u))| \geqslant C_{\tilde{K}}. \tag{5.4}$$

- for all $u \in \mathbb{R}^d$,

$$|\Lambda_{\mathbf{X}}(u) - u| \leqslant M_K. \tag{5.5}$$

Step 1

Lemma 1 (inspired by continuous version (J.Fontbona & F.Panloup)

Let $K > 0$ and $\mu := \mathcal{N}(0, I_d)$. Under $(\mathbf{H_2})$, there exists $\tilde{K} > 0$, such that for all $(x, x', y, y') \in B(0, K)^4$, we can build (Z_1, Z_2) such that

(i) $\mathcal{L}(Z_1) = \mathcal{L}(Z_2) = \mu$,

(ii) there exists $\delta_{\tilde{K}} > 0$ such that

$$\mathbb{P}(F(x, Z_1 + y) = F(x', Z_2 + y')) \geqslant \delta_{\tilde{K}} > 0 \qquad (5.6)$$

(iii) there exists $M_K > 0$ such that

$$\mathbb{P}(|Z_2 - Z_1| \leqslant M_K) = 1. \qquad (5.7)$$

Step 3

Proposition (Calibration of Step 3 duration)

Assume $(\mathbf{H_1})$ and $(\mathbf{H_2})$. Let $\alpha \in \left(\frac{1}{2} \vee \left(\frac{3}{2} - \beta\right), \rho\right)$. Assume that for all $j \geqslant 1$,

$$\Delta t_3^{(j)} = t_* \varsigma^j 2^{\theta \ell_j^*} \text{ with } \theta > (2(\rho - \alpha))^{-1}$$

where $\varsigma > 1$ is arbitrary. Then, for all $K > 0$, there exists a choice of t_* such that, for all $j \geqslant 0$,

$$\mathbb{P}(\Omega^1_{\alpha,\tau_j} | \{\tau_j < +\infty\}) = 1.$$

Recall : Ω^1_{α,τ_j} corresponds to

$$\forall n \geqslant 0, \quad \left| \sum_{k=n+1}^{+\infty} a_k g_{\tau_j+n-k} \right| \leqslant (n+1)^{-\alpha}$$